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## Bloch oscillations of strongly interacting Bose atoms

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**Abstract.** – We analyse Bloch oscillations (i.e., oscillations induced by a static force) of strongly interacting Bose atoms *beyond* the hard-core bosons model. It is shown that residual interactions between the atoms modulate Bloch oscillations, where the type of modulations depends on the magnitude of the static force.

1. Recently much attention has been paid to the transport of cold atoms in optical lattices [1–12]. One of these problems is Bloch oscillations (BO) of atoms in a static, for example, gravitational field (see review [6] and references therein). Besides their own interest, these studies pursue the aim of realizing (and understanding) the ordinary and super-conductivity of cold atoms [8, 12].

The present work is devoted to BO of Bose atoms in deep anisotropic optical lattices, where the system can be described by the 1D Bose-Hubbard model

$$\widehat{H} = \widehat{H}_0 + Fd \sum_{l} l \widehat{n}_l , \qquad (1)$$

$$\hat{H}_0 = -\frac{J}{2} \sum_{l} \left( \hat{a}_{l+1}^{\dagger} \hat{a}_l + h.c. \right) + \frac{W}{2} \sum_{l} \hat{n}_l (\hat{n}_l - 1) . \tag{2}$$

In Eqs. (1-2)  $\hat{a}_l^{\dagger}$  and  $\hat{a}_l$  are the bosonic creation and annihilation operators,  $\hat{n}_l = \hat{a}_l^{\dagger} \hat{a}_l$  the number operator, J the hopping matrix element, W the on-site interaction energy, d the lattice period, and F the magnitude of static force. For vanishing interactions the static force in (1) induces coherent, perfectly periodic oscillations of the atoms with the Bloch frequency  $\omega_B = Fd/\hbar$ . Then the main question is of how non-zero interactions do affect these oscillations.

For a weak to moderate interaction constant,  $W \leq J$ , BO of Bose atoms in 1D lattices were analysed in Ref. [4]. It was found that atom-atom interactions destroy the coherence in the system and BO typically decay within a few Bloch periods. (An exception is the case of strong forcing,  $Fd \gg J$ , where BO of interacting atoms are quasiperiodic [5].) One might naively expect that stronger interactions would cause an even faster decay of BO. This is, however,

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not the case because for  $W\gg J$  the system approaches the hard-core bosons (HC-bosons) limit,

$$\hat{H}_{HC} = -\frac{J}{2} \sum_{l} \left( \hat{a}_{l+1}^{\dagger} \hat{a}_{l} + h.c. \right) ,$$
 (3)

where the additional constraints  $(\hat{a}_l)^2 = (\hat{a}_l^{\dagger})^2 = 0$  prohibit double occupancy of a single well. The system (3) is completely integrable and its Hamiltonian can be diagonalised by using the canonical transformation  $\hat{b}_k = (1/\sqrt{L}) \sum_l \exp(i2\pi k l/L) \hat{a}_l$ , i.e., by going from the Wannier to Bloch basis. Then it is easy to show that BO of HC-bosons do not decay [see Eq. (9) below].

The above result refers to the formal limit  $W \to \infty$ . In practice, however, one deals with a finite interaction constant, which implies the presence of residual interactions between Bose atoms. It is the primary aim of this work to study the effect of these residual interactions on the Bloch dynamics of strongly interacting Bose atoms.

2. We begin with the effective Hamiltonian for strongly interacting Bose atoms(1). Assuming periodic boundary conditions the Hilbert space  $\mathcal{H}_{HC}$  of the system (3) is spanned by the Fock states  $|\mathbf{n}\rangle = |n_1, \dots, n_L\rangle$  with the occupation numbers  $n_l$  being ether zero or unity. To take the residual interactions into account we extend  $\mathcal{H}_{HC}$  by including the Fock states with  $n_l = 2$ ,

$$\hat{H}^{(1+2)} = \begin{vmatrix} \hat{H}^{(2)} & \hat{V} \\ \hat{V}^{\dagger} & \hat{H}_{HC} \end{vmatrix} . \tag{4}$$

In the Hamiltonian (4) the coupling matrix  $\widehat{V}$  is proportional to J and  $\widehat{H}^{(2)}|\mathbf{n}\rangle = W|\mathbf{n}\rangle$ . Using ordinary perturbation theory in the parameter  $J/W \ll 1$  and projecting the eigenstates of (4) back to the Hilbert space  $\mathcal{H}_{HC}$ , we end up with the following effective Hamiltonian,

$$\hat{H}_{SC} = -\frac{J}{2} \sum_{l} \left( \hat{a}_{l+1}^{\dagger} \hat{a}_{l} + h.c. \right)$$

$$-\frac{J^{2}}{W} \sum_{l} \hat{n}_{l+1} \hat{n}_{l} - \frac{J^{2}}{2W} \left( \sum_{l} \hat{a}_{l+1}^{\dagger} \hat{n}_{l} \hat{a}_{l-1} + h.c. \right) . \tag{5}$$

For finite W the 'soft-core bosons' (SC-bosons) Hamiltonian (5) approximates the low-energy spectrum of Bose atoms much better than the HC-bosons Hamiltonian (3). As an example, Fig. 1 compares the low-energy spectrum of the system (2) with the spectrum of the SC-bosons Hamiltonian for J/W=0.1 where, to facilitate the comparison, we parameterised the creation and annihilation operators by the phase  $\theta$ :  $\hat{a}_l^{\dagger} \rightarrow \hat{a}_l^{\dagger} \exp(-il\theta)$ ,  $\hat{a}_l \rightarrow \hat{a}_l \exp(il\theta)$ . (The reason for this parameterisation will become clear in a moment.) Note that some of the level crossings in Fig. 1 are actually avoided crossings. We found, however, that in the relevant region J/W < 0.1 the gaps of the avoided crossings can be safely neglected. This result also means that the single-particle quasimomentum numbers  $k_i$ , i=1,N, are still good quantum numbers to label the eigenstates of the Hamiltonian (5).

3. We proceed with the Bloch dynamics. Using a gauge transformation the static forcing can be presented as a periodic driving of the system with the Bloch frequency  $\omega_B$ ,

$$\widehat{H}_{SC}(t) = \widehat{H}_{HC}(t) + \widehat{H}_{int}(t) , \qquad (6)$$

 $<sup>(^{1})</sup>$ The derivation of the effective Hamiltonian presented in this work is similar to that for the effective Hamiltonian of composed bosons in the frame of the negative-u Fermi-Hubbard model (see, Ref. [13], for example).

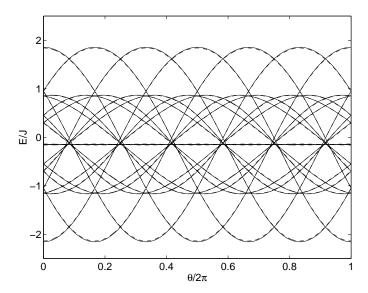


Fig. 1 – Spectrum of the SC-bosons Hamiltonian (5) parameterised by the phase  $\theta$  (solid lines) as compared to the low-energy spectrum of the Hamiltonian (2) (dashed lines). The parameters are L=6, N=3, and W=10J.

$$\hat{H}_{HC}(t) = -\frac{J}{2} \sum_{l} \left( e^{-i\omega_{B}t} \hat{a}_{l+1}^{\dagger} \hat{a}_{l} + h.c. \right) ,$$

$$\hat{H}_{int}(t) = -\frac{J^{2}}{W} \sum_{l} \hat{n}_{l+1} \hat{n}_{l} - \frac{J^{2}}{2W} \sum_{l} \left( e^{-i2\omega_{B}t} \hat{a}_{l+1}^{\dagger} \hat{n}_{l} \hat{a}_{l-1} + h.c. \right) .$$

Then the dynamics of the system is conveniently described by the evolution operator over one Bloch period  $\hat{U}_{SC}(T_B)$ , where

$$\widehat{U}_{SC}(t) = \widehat{\exp}\left[-\frac{i}{\hbar} \int_0^t \widehat{H}_{SC}(t)dt\right]$$
 (7)

and the hat over the exponent sign denotes the time odering. For  $W = \infty$  (the HC-bosons limit) the explicite form of the evolution operator is given by

$$\widehat{U}_{HC}(t) = \widehat{T}^{\dagger} \widehat{D}(t) \widehat{T} , \qquad (8)$$

where the unitary operator  $\widehat{T}$  represents the transformation from the Wannier basis  $|\mathbf{n}\rangle = |n_1, \dots, n_L\rangle$  to the Bloch basis  $|\mathbf{k}\rangle = |k_1, \dots, k_L\rangle$  (2) and the matrix of the operator  $\widehat{D}(t)$  is diagonal in the Bloch basis,

$$\langle \mathbf{k} | D(t) | \mathbf{k} \rangle = \exp \left[ -i \frac{J}{\hbar} \int_0^t \sum_{i=1}^N \cos \left( \frac{2\pi k_i}{L} - \omega_B t \right) dt \right] .$$
 (9)

From (8-9) it follows immediately that  $\widehat{U}_{HC}(T_B) = \widehat{I}$  and, hence, BO of HC-bosons are perfectly periodic. In contrast, for SC-bosons the Bloch evolution operator  $\widehat{U}_{SC}(T_B)$  differs from the identity operator and, thus, BO are generally not periodic.

<sup>(2)</sup> The elements of T are given by the determinant of the  $N \times N$  matrix A with  $A_{i,j} = L^{-1/2} \exp(ik_i l_j/L)$ , where  $k_i$  and  $l_j$  are the occupied orbitals.

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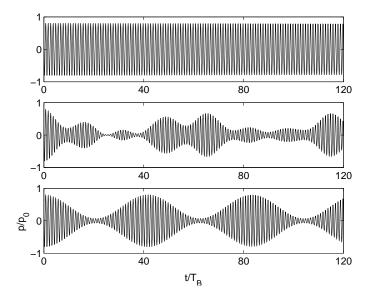


Fig. 2 – Normalised mean momentum  $(p(t) \to p(t)/p_0, p_0 = NJMd/\hbar)$  of strongly interacting Bose atoms for weak (Fd/J = 0.1, upper panel), moderate (Fd/J = 1 middle panel), and strong (Fd/J = 4, lower panel) forcing. The other parameters are L = 8, N = 3, and W = 10J.

Addressing the dynamics of interacting Bose atoms one has to distinguish between strong,  $Fd \gg J$ , and weak,  $Fd \ll J$ , forces. We discuss a weak force first. In this case the phase  $\theta = \omega_B t$  in Eq. (6) can be considered as a slowly varying parameter and, hence, the system adiabatically follows the instantaneous levels shown in Fig. 1. Then the matrix of the evolution operator  $\widehat{U}_{SC}(T_B)$  is diagonal in the eigen-basis of the Hamiltonian  $\widehat{H}_{SC}$ . For  $J/W \ll 1$  this basis practically coincides with the eigen-basis of  $\widehat{H}_{HC}$ , i.e., with the quasimomentum Fock states  $|\mathbf{k}\rangle$ . Thus

$$\langle \mathbf{k} | \hat{U}_{SC}(T_B) | \mathbf{k} \rangle = \exp[-i\Theta(\mathbf{k})],$$
 (10)

where  $\Theta(\mathbf{k}) = \frac{1}{\hbar} \int_0^{T_B} E_{\mathbf{k}}(t) dt$  are the phases acquired by the states during the time evolution. Since the energy differences between the levels of the HC-bosons and SC-bosons Hamiltonians are essentially given by the diagonal elements of  $\widehat{H}_{int}(t)$  in the Bloch basis, we finally have

$$\Theta(\mathbf{k}) = -\frac{2\pi J^2}{WFd} \langle \mathbf{k} | \sum_{l} \hat{n}_{l+1} \hat{n}_{l} | \mathbf{k} \rangle . \tag{11}$$

Assuming the experimentally relevant situation where the system is initially in its ground state, the result (10-11) means that BO of SC-bosons are periodic in the limit of small F. This regime of the system's dynamics is illustrated in the upper panel of Fig. 2, showing the mean atomic momentum of N=3 atoms in L=8 wells (periodic boundary conditions) for J/W=0.1 and Fd/J=0.1.

When the static force is increased, the adiabatic approximation breaks down and the Bloch dynamics of the atoms becomes rather complicated. An example is given in the middle panel in Fig. 2, where the static force Fd/J = 1. It is seen that the mean momentum p(t) undergoes a complex quasiperiodic process, involving many different frequencies. These

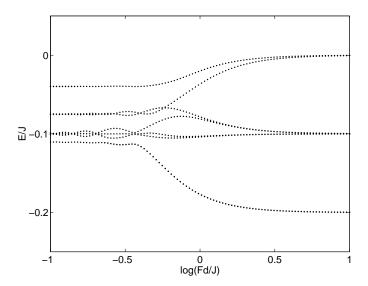


Fig. 3 – Quasienergy spectrum of the system for different magnitudes of the static force. The system parameters are L=8, N=3, and W=10J.

frequencies are obviously given by the quasienergies E, which are defined by the eigen-phases  $\Theta$  of the evolution operator,

$$\widehat{U}_{SC}(T_B)|\Phi\rangle = \exp(-i\Theta)|\Phi\rangle , \qquad (12)$$

through the relation  $\Theta = ET_B/\hbar = 2\pi E/\hbar\omega_B$ . For the considered system of N=3 atoms in L=8 wells the quasienergies E are plotted in Fig. 3 as the functions of F, the strength of the static force. The transition from the adiabatic to the non-adiabatic regime, reflected in the removed degeneracy between the quasienergy levels, is clearly seen(3). It is also an appropriate place here to mention the particle-hole symmetry of the system. In terms of the evolution operator it means that the quasienergy spectum of N=3 atoms in L=8 wells coincides (up to an irrelevant global shift) with the spectrum of N=5 atoms in L=8 wells.

The further increase of F simplifies the Bloch dynamics again, as can already be concluded from the structure of the quasienergy spectrum for  $Fd \gg J$  (see Fig. 3). Let us show that in this region BO of SC-bosons are periodically modulated with the period

$$T_W = \frac{2\pi\hbar W}{J^2} = \frac{W}{J} \frac{Fd}{J} T_B \tag{13}$$

(see lower panel in Fig. 2). Indeed, for strong forcing the Bloch period  $T_B$  is the shortest time scale in the system and, hence, one can separate the slow (modulation) and fast (BO) dynamics. Introducing the 'slow' wave-function  $|\widetilde{\Psi}(t)\rangle$ ,  $|\Psi(t)\rangle = \widehat{U}_{HC}(t)|\widetilde{\Psi}(t)\rangle$ , the Schrödinger equation reads

$$i\hbar \frac{\partial |\widetilde{\Psi}\rangle}{\partial t} = \widehat{H}_{slow} |\widetilde{\Psi}\rangle ,$$
 (14)

$$\widehat{H}_{slow} = \frac{1}{T_B} \int_0^{T_B} \widehat{U}_{HC}^{\dagger}(t) \widehat{H}_{int}(t) \widehat{U}_{HC}(t) dt , \qquad (15)$$

<sup>(3)</sup>Still, since the global quasimomentum is conserved, each of the quasienergy levels is L-fold degenerate.

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where  $\widehat{U}_{HC}(t)$  is given in Eqs. (8-9). Noting that for  $Fd \gg J$  the operator  $\widehat{D}(t) \approx \widehat{I}$ , we conclude that the matrix of the evolution operator  $\widehat{U}_{SC}(T_B)$  is diagonal in the Wannier basis,

$$\langle \mathbf{n}|\hat{U}_{SC}(T_B)|\mathbf{n}\rangle = \exp[-i\Theta(\mathbf{n})],$$
 (16)

$$\Theta(\mathbf{n}) = -\frac{2\pi J^2}{WFd} \langle \mathbf{n} | \sum_{l} \hat{n}_{l+1} \hat{n}_{l} | \mathbf{n} \rangle .$$
 (17)

Furthermore, because the quantities  $\langle \mathbf{n} | \sum_{l} \hat{n}_{l+1} \hat{n}_{l} | \mathbf{n} \rangle$  are integers, the relation (16) ensures perfectly periodic modulations of the BO,

$$p(t) = [A + B\cos(2\pi t/T_W)]\sin(\omega_B t). \tag{18}$$

The constants A and B in Eq. (18) depend on the filling factor  $\bar{n} = N/L$  and one gets the most pronounced modulations for  $\bar{n} \sim 1/2$  (<sup>4</sup>). It is also worth of mentioning that the above analysis implicitly assumes Fd < W. Indeed, for  $Fd \approx W$  the static forcing resonantly couples the Fock states belonging to  $\mathcal{H}_{HC}$  to the states with  $n_l = 2$  and our approach (based on ordinary perturbation theory) is not valid. For a particular case of the Mott-insulator initial state ( $\bar{n} = 1$ ) the dynamical responce of the system to the resonant static force was analysed in Refs. [15–17].

4. In conclusion, we have studied the effect of residual atom-atom interactions on the Bloch dynamics of strongly interacting Bose atoms. The characteristic energy of these interactions is given by the ratio of the squared hopping matrix element J and the microscopic interaction constant W, and is usually neglected when one discusses the equilibrium properties of the system (i.e., one uses the hard-core bosons model [18,19]). The effect of residual interactions, however, accumulates in time when it concerns the dynamics. In particular, it has been shown that they may modulate BO of Bose atoms. We have provided a complete description of the dynamical regimes of the system in terms of the Floquet-Bloch evolution operator (evolution operator over one Bloch period  $T_B = 2\pi\hbar/Fd$ ). The matrix of this operator is found to be diagonal in the Bloch basis in the case of a weak forcing,  $Fd \ll J$ , diagonal in the Wannier basis in the opposite case  $Fd \gg J$ , and non-diagonal in either of basis for a moderate forcing,  $Fd \sim J$ . As the result, BO of the strongly interacting Bose atoms are periodic for a weak static force, periodically modulated with the period  $T_W = 2\pi\hbar W/J^2$  for a strong static force, and involve multiple time scales for a moderate static force.

It is interesting to estimate the modulation period  $T_W$  in the typical laboratory experiment on BO of Bose atoms in the gravitational field. Taking, as an example, rubidium atoms in a 3D optical lattice with  $d=0.405\mu \text{m}$  ( $E_R/2\pi\hbar=3558\text{Hz}$ ),  $V_z=6E_R$ , and  $V_\perp=30E_R$ , we have  $J^2/W=0.1^2/1.3\approx 0.008E_R$  and the Stark energy  $Fd=0.24E_R$ . Thus the modulation period  $T_W=30T_B=35\text{ms}$ . This time is easily accessible in the present day experiments. It should be noted, however, that the results of the present work refer to a finite lattice with periodic boundary conditions, while in practice one typically uses a harmonic confinement of the system. The analysis of the Bloch dynamics of Bose atoms for these different boundary conditions will be the subject of a separate paper.

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<sup>(4)</sup>The functional dependence (18) also holds for BO of two-component Fermi gas (repulsive interactions) in the limit of strong forcing  $Fd \gg J$  [14].

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